Maximal covers of chains of prime ideals

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Abstract

Suppose $f : S \to R$ is a ring homomorphism such that $f[S]$ is contained in the center of R. We study the connections between chains in $\mathrm{Spec}(S)$ and chains in $\mathrm{Spec}(R)$. We focus on the properties LO (lying over), INC (incomparability), GD (going down), GU (going up) and SGB (strong going between). We provide a sufficient condition for every maximal chain in $\mathrm{Spec}(R)$ to cover a maximal chain in $\mathrm{Spec}(S)$. We prove some necessary and sufficient conditions for f to satisfy each of the properties GD, GU and SGB, in terms of maximal $\mathcal{D}$-chains, where $\mathcal{D} \subseteq \mathrm{Spec}(S)$ is a nonempty chain. We show that if f satisfies all of the properties above, then every maximal $\mathcal{D}$-chain is a perfect maximal cover of $\mathcal{D}$. Our main result is Corollary 2.11, in which we give equivalent conditions for the following property: for every chain $\mathcal{D} \subseteq \mathrm{Spec}(S)$ and for every maximal $\mathcal{D}$-chain $\mathcal{C} \subseteq \mathrm{S}pec(R)$, $\mathcal{C}$ and $\mathcal{D}$ are of the same cardinality.

Introduction

All rings considered in this paper are nontrivial rings with identity; homomorphisms need not be unitary, unless otherwise stated. All chains considered are chains with respect to containment. The symbol $\subset$ means proper inclusion and the symbol $\subseteq$ means inclusion or equality. For a ring R we denote the center of R by $Z(R)$.

We start with some background on the notions we consider.

In his highly respected paper from 1937, in which Krull (see [Kr]) proved his basic theorems regarding the behavior of prime ideals under integral extensions, and defined the properties LO, GU, GD and INC, he proposed the following question: assuming $S \subseteq R$ are integral domains with R integral over S and S integrally closed; do adjacent prime ideals in R contract to adjacent prime ideals in S? In 1972 Kaplansky (see [Ka]) answered the question negatively. In 1977, Ratliff (see [Ra]) made another step; he defined and studied the notion GB (going between): let $S \subseteq R$ be commutative rings. $S \subseteq R$ is said to satisfy GB if whenever $Q_1 \subset Q_2$ are prime ideals of R and there exists a prime ideal $Q_1 \cap S \subset P' \subset Q_2 \cap S$, then there exists a prime ideal Q' in R such that $Q_1 \subset Q' \subset Q_2$. Later, in 2003, G. Picavet (see [Pi]) introduced the notion SGB (strong going between), to be presented later.

In 2003, Kang and Oh (cf. [KO]) defined the notion of an SCLO extension: let $S \subseteq R$ be commutative rings. $S \subseteq R$ is called an SCLO extension if for every chain of prime ideals $\mathcal{D}$ of S with an initial element P and Q a prime ideal of R lying over P, there exists a chain of prime ideals $\mathcal{C}$ of R lying over $\mathcal{D}$ whose initial element is Q. They proved that $S \subseteq R$ is an SCLO extension iff it is a GU extension (cf [KO, Corollary 12]). In particular, if $S \subseteq R$ is a GU extension then for every chain of prime ideals $\mathcal{D}$ of S there exists a chain of prime ideals $\mathcal{C}$ of R covering $\mathcal{D}$. We note that when considering a unitary homomorphism $f : S \to R$, some authors call f a chain morphism if every chain in spec(S) can be covered by a chain in Spec(R). Also, some authors call an SCLO extension a GGU (generalized going up) extension. It follows that, in the commutative case, GU implies GGU for unitary ring homomorphisms. In 2005 Dobbs and Hetzel (cf. [DH]) proved that, in the commutative case, GD implies GGD for unitary ring homomorphisms. GGD is defined analogously to GGU.

We consider a more general setting. Namely, *In this paper S and R are rings (not necessarily commutative) and $f : S \to R$ is a homomorphism (not necessarily unitary) such that $f[S] \subseteq Z(R)$.* We study the connections between chains of prime ideals of S and chains of prime ideals of R.

For $I \triangleleft R$ and $J \triangleleft S$ we say that I is lying over J if $J = f^{-1}[I]$. Note that if Q is a prime ideal of R then $f^{-1}[Q]$ is a prime ideal of S or $f^{-1}[Q] = S$. It is not difficult to see that f is unitary if and only if for all $Q \in \mathrm{Spec}(R)$, $f^{-1}[Q] \in \mathrm{Spec}(S)$. Indeed, ($\Rightarrow$) is easy to check and as for ($\Leftarrow$), if f is not unitary then $f(1)$ is a central idempotent$\neq 1$. Thus, there exists a prime ideal $Q \in \mathrm{Spec}(R)$ containing $f(1)$; clearly, $f^{-1}[Q] = S$.

For the reader's convenience, we define now the five basic properties we consider.

We say that f satisfies LO (lying over) if for all $P \in \mathrm{S}pec(S)$ there exists $Q \in \mathrm{S}pec(R)$ lying over P.

We say that f satisfies GD (going down) if for any $P_1 \subset P_2$ in $\mathrm{Spec}(S)$ and every $Q_2 \in \mathrm{Spec}(R)$ lying over P_2, there exists $Q_1 \subset Q_2$ in $\mathrm{Spec}(R)$ lying over P_1.

We say that f satisfies GU (going up) if for any $P_1 \subset P_2$ in $\mathrm{Spec}(S)$ and every $Q_1 \in \mathrm{Spec}(R)$ lying over P_1, there exists $Q_1 \subset Q_2$ in $\mathrm{Spec}(R)$ lying over P_2.

We say that f satisfies SGB (strong going between) if for any $P_1 \subset P_2 \subset P_3$ in $\mathrm{Spec}(S)$ and every $Q_1 \subset Q_3$ in $\mathrm{Spec}(R)$ such that Q_1 is lying over P_1 and Q_3 is lying over P_3, there exists $Q_1 \subset Q_2 \subset Q_3$ in $\mathrm{Spec}(R)$ lying over P_2.

We note that the "standard" definition of INC is as follows: f is said to satisfy INC (incomparability) if whenever $Q_1 \subset Q_2$ in $\mathrm{Spec}(R)$, we have $f^{-1}[Q_1] \subset f^{-1}[Q_2]$. However, in order for us to be able to talk about non-unitary homomorphisms we make the following modification and define INC as follows: we say that f satisfies INC if whenever $Q_1 \subset Q_2$ in $\mathrm{Spec}(R)$ and

$f^{-1}[Q_2] \neq S$, we have $f^{-1}[Q_1] \subset f^{-1}[Q_2]$.

We note that the case in which R is an algebra over S can be considered as a special case (of the case we consider). In particular, the case in which $S \subseteq R$ are rings and $S \subseteq Z(R)$ (and thus S is a commutative ring) can be considered as a special case. Also, in this special case, one can show that, as in the commutative case, GU implies LO. However, in our case, GU does not necessarily imply LO. As an easy example, let $p \in \mathbb{N}$ be a prime number and let $2 \leq n \in \mathbb{N}$ such that n is not a power of p. Let $f : \mathbb{Z}_{np} \to \mathbb{Z}_p$ be the homomorphism defined by $x \bmod (np) \to x \bmod (p)$. Then f satisfies GU (in a trivial way, since k-dim $\mathbb{Z}_{np} = 0$) but there is no prime ideal in $\mathbb{Z}_p$ lying over $q\mathbb{Z}_{np}$, where $p \neq q \in \mathbb{N}$ is any prime number dividing n.

1   Maximal chains in $\mathrm{Spec}(R)$ that cover maximal chains in $\mathrm{Spec}(S)$

In this section we prove that every maximal chain of prime ideals of R covers a maximal chain of prime ideals of S, under the assumptions GU, GD and SGB. We also present an example from quasi-valuation theory.

We start with some basic definitions.

Definition 1.1. Let $\mathcal{D}$ denote a set of prime ideals of S. We say that a chain $\mathcal{C}$ of prime ideals of R is a $\mathcal{D}$-*chain* if $f^{-1}[Q] \in \mathcal{D}$ for every $Q \in \mathcal{C}$.

Definition 1.2. Let $\mathcal{D} \subseteq \mathrm{S}pec(S)$ be a chain of prime ideals and let $\mathcal{C} \subseteq \mathrm{S}pec(R)$ be a $\mathcal{D}$-chain. We say that $\mathcal{C}$ is a *cover* of $\mathcal{D}$ (or that $\mathcal{C}$ covers $\mathcal{D}$) if for all $P \in \mathcal{D}$ there exists $Q \in \mathcal{C}$ lying over P; i.e. the map $Q \to f^{-1}[Q]$ from $\mathcal{C}$ to $\mathcal{D}$ is surjective. We say that $\mathcal{C}$ is a *perfect cover* of $\mathcal{D}$ if $\mathcal{C}$ is a cover of $\mathcal{D}$ and the map $Q \to f^{-1}[Q]$ from $\mathcal{C}$ to $\mathcal{D}$ is injective. In other words, $\mathcal{C}$ is a perfect cover of $\mathcal{D}$ if the map $Q \to f^{-1}[Q]$ from $\mathcal{C}$ to $\mathcal{D}$ is bijective.

The following lemma is well known.

Lemma 1.3. *Let R be a ring. Let $\alpha \in Z(R)$, $r \in R$ and $Q \in Spec(R)$. If $\alpha r \in Q$ then $\alpha \in Q$ or $r \in Q$.*

Proof. $\alpha R r = R \alpha r \subseteq Q$. Thus $\alpha \in Q$ or $r \in Q$.   □

We briefly recall now the notion of a weak zero divisor, introduced in [BLM]. For a ring R, $a \in R$ is called a weak zero-divisor if there are $r_1, r_2 \in R$ with $r_1 a r_2 = 0$ and $r_1 r_2 \neq 0$. In [BLM] it is shown, that in any ring R, the elements of a minimal prime ideal are weak zero-divisors. Explicitly, the following is proven (see [BLM, Theorem 2.2]).

Lemma 1.4. *Let R be a ring and let $Q \in Spec(R)$ denote a minimal prime of R. Then for every $q \in Q$, q is a weak zero divisor.*

It will be more convenient for us to use the following version of the previous Lemma.

Lemma 1.5. *Let R be a ring, let $I \lhd R$ and let $Q \in Spec(R)$ denote a minimal prime ideal over I. Then for every $q \in Q$ there exist $r_1, r_2 \in R$ such that $r_1 q r_2 \in I$ and $r_1 r_2 \notin I$.*

We return now to our discussion. But before proving the next proposition, we note that R is not necessarily commutative. Therefore, a union of prime ideals of a chain in $\mathrm{Spec}(R)$ is not necessarily a prime of R.

Proposition 1.6. *Let $\mathcal{C} \subseteq Spec(R)$ denote a chain of prime ideals of R. Let $\bigcup_{Q \in \mathcal{C}} Q \subseteq Q' \in Spec(R)$ denote a minimal prime ideal over $\bigcup_{Q \in \mathcal{C}} Q$. Then $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q] = f^{-1}[Q']$. In particular, if f is unitary then $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]$ is a prime ideal of S.*

Proof. Assume to the contrary that $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q] \subset f^{-1}[Q']$ and let $s \in f^{-1}[Q'] \setminus f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]$. By Lemma 1.5, there exist $r_1, r_2 \in R$ such that $r_1 f(s) r_2 \in \bigcup_{Q \in \mathcal{C}} Q$ and $r_1 r_2 \notin \bigcup_{Q \in \mathcal{C}} Q$. However, $f(s) \in f[S] \subseteq Z(R)$ and thus $f(s) r_1 r_2 = r_1 f(s) r_2 \in \bigcup_{Q \in \mathcal{C}} Q$. So, there exists $Q \in \mathcal{C}$ such that $f(s) r_1 r_2 \in Q$. Note that $f(s) \notin \bigcup_{Q \in \mathcal{C}} Q$; hence, by Lemma 1.3, $r_1 r_2 \in Q$. I.e., $r_1 r_2 \in \bigcup_{Q \in \mathcal{C}} Q$, a contradiction $\qquad\qquad\square$

So, a union of prime ideals of a chain in $\mathrm{Spec}(R)$ and a minimal prime ideal over it, are lying over the same prime ideal of S (or over S). Now, given an ideal I of R that is contained in an arbitrary prime ideal Q_2 of R, we prove the existence of a minimal prime ideal over I, contained in Q_2.

Lemma 1.7. *Let I be an ideal of R and let Q_2 be any prime ideal of R containing I. Then there exists $I \subseteq Q' \subseteq Q_2$, a minimal prime ideal over I. In particular, if J is any ideal of R then there exists a minimal prime ideal over J.*

Proof. By Zorn's Lemma there exists a maximal chain of prime ideals, say $\mathcal{C}'$, between I and Q_2. So, $I \subseteq \bigcap_{Q \in \mathcal{C}'} Q \subseteq Q_2$ is a minimal prime ideal over I. The last assertion is clear. $\qquad\qquad\square$

So, we have managed to prove the following: let $\mathcal{C} \subseteq \mathrm{Spec}(R)$ denote a chain of prime ideals of R and let Q_2 be a prime ideal of R containing $\bigcup_{Q \in \mathcal{C}} Q$. Then there exists a prime ideal $\bigcup_{Q \in \mathcal{C}} Q \subseteq Q' \subseteq Q_2$ such that Q' is lying over $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]$, which is a prime ideal of S or equals S. One may wonder if this fact can be generalized as follows: let I be an ideal of R lying over a prime ideal of S and let $I \subseteq Q_2$ be any prime ideal of R. Must there be a prime ideal $I \subseteq Q' \subseteq Q_2$ in R such that Q' is lying over the prime ideal

$f^{-1}[I]$? The answer to this question is: "no", even in the case where $S \subseteq R$ are commutative rings and R is integral over S. As shown in the following example.

Example 1.8. Let $S \subseteq R$ be rings such that R satisfies LO over S but not GD (in our terminology, assume that the map $f : S \to R$, defined by $f(s) = s$ for all $s \in S$, satisfies LO but not GD). Let $P_1 \subset P_2$ be prime ideals of S and let Q_2 be a prime ideal of R lying over P_2 such that there is no prime ideal $Q_1 \subset Q_2$ lying over P_1. Let Q_3 be a prime ideal of R lying over P_1. Denote $I = Q_3 \cap Q_2$ and note that I is clearly not a prime ideal of R; then, I is an ideal of R lying over the prime ideal P_1 and $I \subset Q_2$, but there is no prime ideal $I \subseteq Q' \subset Q_2$ of R lying over P_1.

The following two remarks are obvious.

Remark 1.9. Let $\mathcal{C}$ denote a chain of prime ideals of R. Then the map $Q \to f^{-1}[Q]$ from $\mathcal{C}$ to $\mathrm{S}pec(S)$ is order preserving.

Remark 1.10. Assume that f satisfies INC. Let $\mathcal{C}$ denote a chain of prime ideals of R and let $\mathcal{C}' = \{Q \in \mathcal{C} \mid f^{-1}[Q] \neq S\}$. Then the map $Q \to f^{-1}[Q]$ from $\mathcal{C}$ to $\mathrm{Spec}(S)$ is injective.

For the reader convenience, in order to establish the following results in a clearer way, we briefly present here the notion of cuts.

Let T denote a totally ordered set. A subset X of T is called *initial* (resp. *final*) if for every $\gamma \in X$ and $\alpha \in T$, if $\alpha \leq \gamma$ (resp. $\alpha \geq \gamma$), then $\alpha \in X$. A cut $\mathcal{A} = (\mathcal{A}^L, \mathcal{A}^R)$ of T is a partition of T into two subsets $\mathcal{A}^L$ and $\mathcal{A}^R$, such that, for every $\alpha \in \mathcal{A}^L$ and $\beta \in \mathcal{A}^R$, $\alpha < \beta$. To define a cut, one often writes $\mathcal{A}^L = X$, meaning that $\mathcal{A}$ is defined as $(X, T \setminus X)$ when X is an initial subset of T. For more information about cuts see, for example, [FKK] or [Weh].

Lemma 1.11. *Let* $\mathcal{C} = \{X_\alpha\}_{\alpha \in I}$ *be a chain of subsets of R. Let* $\mathcal{D} = \{f^{-1}[X_\alpha]\}_{\alpha \in I}$ *and let* $Y \subseteq S$, $Y \notin \mathcal{D}$. *Let* $A = \{X \in \mathcal{C} \mid f^{-1}[X] \subset Y\}$ *and* $B = \{X \in \mathcal{C} \mid f^{-1}[X] \supset Y\}$. *If* $X \in A$ *and* $X' \in \mathcal{C} \setminus A$ *then* $X \subset X'$. *If* $X \in B$ *and* $X' \in \mathcal{C} \setminus B$ *then* $X \supset X'$. *In particular, assuming* $A, B \neq \emptyset$ *and denoting* $X_1 = \bigcup_{X \in A} X$ *and* $X_2 = \bigcap_{X \in B} X$, *one has* $X_1 \subseteq X_2$.

Proof. Clearly, A is an initial subset of $\mathcal{C}$ and B is a final subset of $\mathcal{C}$. The assertions are now obvious.

$\square$

We shall freely use Lemma 1.11 without reference.

We note that a maximal chain of prime ideals exists by Zorn's Lemma and is nonempty, since every ring has a maximal ideal. Also note that if $\mathcal{C} = \{Q_\alpha\}_{\alpha \in I}$ is a chain of prime ideals of R then $\mathcal{D} = \{f^{-1}[Q_\alpha]\}_{\alpha \in I} \subseteq \mathrm{S}pec(S) \cup \{S\}$ is a chain; if f is unitary then $\mathcal{D}$ is a chain of prime ideals of S. Our immediate objective (to be reached in theorem 1.14) is to prove

that, under certain assumptions, if $\mathcal{C}$ is a maximal chain in $Spec(R)$ then $\mathcal{D}$ is a maximal chain in $Spec(S)$.

Remark 1.12. Let $\mathcal{C}$ be a maximal chain of prime ideals of R, let $A \neq \emptyset$ be an initial subset of $\mathcal{C}$ and let $B \neq \emptyset$ be a final subset of $\mathcal{C}$. Then $\bigcap_{Q \in B} Q \in \mathcal{C}$. Moreover, if $\bigcup_{Q \in A} Q$ is a prime ideal of R then $\bigcup_{Q \in A} Q \in \mathcal{C}$.

Proof. $\bigcap_{Q \in B} Q \in Spec(R)$ and $\mathcal{C} \cup \{\bigcap_{Q \in B} Q\}$ is a chain; thus $\bigcap_{Q \in B} Q \in \mathcal{C}$. In a similar way, if $\bigcup_{Q \in A} Q$ is prime then $\mathcal{C} \cup \{\bigcup_{Q \in A} Q\}$ is a chain of prime ideals in $Spec(R)$; thus $\bigcup_{Q \in A} Q \in \mathcal{C}$. $\qquad\square$

Lemma 1.13. *Let $\mathcal{C} = \{Q_\alpha\}_{\alpha \in I}$ be a maximal chain of prime ideals of R, $\mathcal{D} = \{f^{-1}[Q_\alpha]\}_{\alpha \in I}$, $P \in Spec(S) \setminus \mathcal{D}$, $A = \{Q \in \mathcal{C} \mid f^{-1}[Q] \subset P\}$ and $B = \{Q \in \mathcal{C} \mid f^{-1}[Q] \supset P\}$. Assume that $A, B \neq \emptyset$ and denote $Q_1 = \bigcup_{Q \in A} Q$ and $Q_2 = \bigcap_{Q \in B} Q$. Then $Q_2 \in B$ and $Q_1 \in A$.*

Proof. We start by proving that $Q_2 \in B$. First note that $f^{-1}[Q_2] \supseteq P$. By Remark 1.12, $Q_2 \in \mathcal{C}$. Since $P \in Spec(S) \setminus \mathcal{D}$, one cannot have $f^{-1}[Q_2] = P$. Thus, $Q_2 \in B$.

We shall prove now that Q_1 is a prime ideal of R. Assume to the contrary and let $Q_3 = \bigcap_{Q \in \mathcal{C} \setminus A} Q$. By Remark 1.12, $Q_3 \in \mathcal{C}$. Since $\mathcal{C}$ is a maximal chain in $Spec(R)$, Q_3 is a minimal prime ideal over Q_1, strictly containing it. Clearly $f^{-1}[Q_1] \subseteq P$ and by Proposition 1.6, $f^{-1}[Q_1] = f^{-1}[Q_3]$. Since $P \in Spec(S) \setminus \mathcal{D}$, one cannot have $f^{-1}[Q_3] = P$. Therefore, $f^{-1}[Q_3] \subset P$, i.e., $Q_3 \in A$, a contradiction (to the fact that Q_3 strictly contains Q_1). So, Q_1 is a prime ideal of R. We conclude by Remark 1.12 that $Q_1 \in \mathcal{C}$. Thus, $Q_1 \in A$. $\qquad\square$

Note that by the previous Lemma, Q_1 is the greatest member of A and Q_2 is the smallest member of B.

Theorem 1.14. *Assume that f is unitary and satisfies GU, GD and SGB. Let $\mathcal{C} = \{Q_\alpha\}_{\alpha \in I}$ be a maximal chain of prime ideals in $Spec(R)$. Then $\mathcal{D} = \{f^{-1}[Q_\alpha]\}_{\alpha \in I}$ is a maximal chain of prime ideals in $Spec(S)$. In other words, every maximal chain in $Spec(R)$ is a cover of some **maximal** chain in $Spec(S)$.*

Proof. First note that since f is unitary, every prime ideal of R is lying over some prime ideal of S. Now, since $\mathcal{C}$ is a maximal chain in $Spec(R)$, $\bigcup_{\alpha \in I} Q_\alpha$ is a maximal ideal of R containing each $Q_\alpha \in \mathcal{C}$ and thus $\bigcup_{\alpha \in I} Q_\alpha \in \mathcal{C}$. In a similar way, since $\bigcap_{\alpha \in I} Q_\alpha$ is a prime ideal of R contained in each $Q_\alpha \in \mathcal{C}$, $\bigcap_{\alpha \in I} Q_\alpha \in \mathcal{C}$. We prove that $\bigcap_{\alpha \in I} Q_\alpha$ is lying over a minimal prime ideal of S. Indeed, $\bigcap_{\alpha \in I} Q_\alpha$ is lying over a prime ideal P of S; assume to the contrary that there exists a prime ideal $P_0 \subset P$. Then by GD, there exists

$Q_0 \subset \bigcap_{\alpha \in I} Q_\alpha$ lying over P_0. Thus, $\mathcal{C} \cup \{Q_0\}$ is chain of prime ideals strictly containing $\mathcal{C}$, a contradiction. Similarly, we prove that $\bigcup_{\alpha \in I} Q_\alpha$ is lying over a maximal ideal of S. Indeed, $\bigcup_{\alpha \in I} Q_\alpha$ is lying over a prime ideal P of S; assume to the contrary that there exists a prime ideal $P \subset P'$. Then by GU, there exists $\bigcup_{\alpha \in I} Q_\alpha \subset Q'$ lying over P'. Thus, $\mathcal{C} \cup \{Q'\}$ is a chain of prime ideals strictly containing $\mathcal{C}$, a contradiction.

We prove now that $\mathcal{D} = \{f^{-1}[Q_\alpha]\}_{\alpha \in I}$ is a maximal chain of prime ideals of S. Assume to the contrary that there exists $P \in \mathrm{S}pec(S)$ such that $\mathcal{D} \cup \{P\}$ is a chain of prime ideals strictly containing $\mathcal{D}$. Denote $A = \{Q \in \mathcal{C} \mid f^{-1}[Q] \subset P\}$ and $B = \{Q \in \mathcal{C} \mid f^{-1}[Q] \supset P\}$. Note that by the previous paragraph P cannot be the greatest element nor the smallest element in $\mathcal{D} \cup \{P\}$; therefore $A, B \neq \emptyset$. Denote $Q_1 = \bigcup_{Q \in A} Q$ and $Q_2 = \bigcap_{Q \in B} Q$ and let $P_1 = f^{-1}[Q_1]$ and $P_2 = f^{-1}[Q_2]$; by Lemma 1.13, $Q_1 \in A$ and $Q_2 \in B$. Hence, $P_1 \subset P \subset P_2$. Therefore, by SGB, there exists a prime ideal $Q_1 \subset Q \subset Q_2$ lying over P. It is easy to see that $\mathcal{C}$ is a disjoint union of A and B since $\mathcal{D} \cup \{P\}$ is a chain. Now, by the definition of Q_1 and Q_2, we get $Q' \subseteq Q_1 \subset Q \subset Q_2 \subseteq Q''$ for every $Q' \in A$ and $Q'' \in \mathcal{C} \setminus A = B$. Thus, $\mathcal{C} \cup \{Q\}$ is a chain strictly containing $\mathcal{C}$, a contradiction to the maximality of $\mathcal{C}$. $\square$

It is easy to see that if f is not unitary then Theorem 1.14 is not valid. As a trivial example, take f as the zero map. Obviously, f satisfies GU, GD and SGB, in a trivial way. Here is a less trivial example: let R be a ring and consider the homomorphism $f : R \to R \oplus R$ sending each $r \in R$ to $(r, 0) \in R \oplus R$. It is easy to see that f satisfies GU, GD and SGB. Now, let $\{Q_\alpha\}_{\alpha \in I}$ be a maximal chain of prime ideals in $\mathrm{S}pec(R)$; then $\mathcal{C} = \{R \oplus Q_\alpha\}_{\alpha \in I}$ is a maximal chain of prime ideals in $\mathrm{S}pec(R \oplus R)$. However $\{f^{-1}[R \oplus Q_\alpha]\}_{\alpha \in I} = \{R\}$, which is clearly not a maximal chain of prime ideals of R.

Corollary 1.15. *Assume that f is unitary and satisfies INC, GU, GD and SGB. Let $\mathcal{C} = \{Q_\alpha\}_{\alpha \in I}$ be a maximal chain of prime ideals in $Spec(R)$. Then $\mathcal{C}$ is a perfect cover of the **maximal** chain $\mathcal{D} = \{f^{-1}[Q_\alpha]\}_{\alpha \in I}$.*

Proof. By Lemma 1.10 and Theorem 1.14. $\square$

Example 1.16. Suppose F is a field with valuation v and valuation ring O_v, A is a finite dimensional $F-$algebra and $R \subseteq A$ is a subring of A lying over O_v. By [Sa1, Theorem 9.34] there exists a quasi-valuation w on RF extending the valuation v, with R as its quasi-valuation ring. By [Sa2], if the value monoid of w is cancellative (for example, when R is finitely generated as a module over O_v, see [Sa2]), then R satisfies LO, INC, GU, GD, and SGB over O_v. Now, let $\mathcal{C} = \{Q_\alpha\}_{\alpha \in I}$ denote **any maximal chain** of prime ideals of R. By Corollary 1.15, the map $Q \to Q \cap O_v$ is a bijective

order preserving correspondence between $\mathcal{C}$ and **the maximal chain** $\mathcal{D} = \{Q_\alpha \cap S\}_{\alpha \in I}$ of prime ideals of O_v, namely, $\mathcal{D} = \mathrm{S}pec(O_v)$. In other words, any maximal chain in $\mathrm{S}pec(R)$ covers the maximal chain $\mathrm{S}pec(O_v)$ in a one-to-one correspondence. In particular, for any chain in $\mathrm{S}pec(O_v)$ there exists a chain in $\mathrm{S}pec(R)$ covering it, in a one-to-one correspondence.

For more information on quasi-valuations see [Sa1], [Sa2] and [Sa3].

2 Maximal $\mathcal{D}$-chains

We shall now study the subject from the opposite point of view: we take $\mathcal{D}$, a chain of prime ideals of S and study $\mathcal{D}$-chains; in particular, maximal $\mathcal{D}$-chains.

we start with the definition of a maximal $\mathcal{D}$-chain.

Definition 2.1. Let $\mathcal{D}$ be a chain of prime ideals of S and let $\mathcal{C}$ be a $\mathcal{D}$-chain. We say that $\mathcal{C}$ is a *maximal $\mathcal{D}$-chain* (not to be confused with a maximal chain) if whenever $\mathcal{C}'$ is a chain of prime ideals of R strictly containing $\mathcal{C}$ then there exists $Q \in \mathcal{C}'$ such that $f^{-1}[Q] \notin \mathcal{D}$. Namely, $\mathcal{C}$ is a $\mathcal{D}$-chain which is maximal with respect to containment.

We shall now prove a basic lemma, the existence of maximal $\mathcal{D}$-chains.

Lemma 2.2. *f satisfies LO if and only if for every nonempty chain $\mathcal{D} \subseteq Spec(S)$, there exists a nonempty maximal $\mathcal{D}$-chain.*

Proof. ($\Rightarrow$) Let $\mathcal{D} \subseteq \mathrm{S}pec(S)$ be a nonempty chain, let $P \in \mathcal{D}$, and let

$$\mathcal{Z} = \{\mathcal{E} \subseteq \mathrm{S}pec(R) \mid \mathcal{E} \text{ is a nonempty } \mathcal{D} \text{-chain}\}.$$

By LO there exists $Q \in \mathrm{S}pec(R)$ such that $f^{-1}[Q] = P$; hence $\{Q\} \in \mathcal{Z}$. Therefore, $\mathcal{Z} \neq \emptyset$. Now, $\mathcal{Z}$ with the partial order of containment satisfies the conditions of Zorn's Lemma and thus there exists $\mathcal{C} \in \mathcal{Z}$ maximal with respect to containment.

($\Leftarrow$) It is obvious. $\square$

Note that one can prove a similar version of ($\Rightarrow$) of Lemma 2.2 without the LO assumption. However, in this case a maximal $\mathcal{D}$-chain might be empty. Also note that similarly, one can prove that for any $\mathcal{D}$-chain $\mathcal{C}'$ there exists a maximal $\mathcal{D}$-chain $\mathcal{C}$ containing $\mathcal{C}'$.

Our immediate goal is to provide a preliminary connection between the properties LO, INC, GU, GD and SGB and maximal $\mathcal{D}$-chains. In order to do that, we define the following definition.

Definition 2.3. Let $n \in \mathbb{N}$. We say that f satisfies the *layer n property* if for every chain $\mathcal{D} \subseteq \mathrm{S}pec(S)$ of cardinality n, every maximal $\mathcal{D}$-chain is of cardinality n.

We shall prove that if f satisfies layers 1, 2 and 3 properties then for every chain $\mathcal{D} \subseteq Spec(S)$ of arbitrary cardinality, every maximal $\mathcal{D}$-chain is of the same cardinality.

The following proposition is seen by an easy inspection.

Proposition 2.4. *1. f satisfies the layer 1 property if and only if f satisfies LO and INC.*

2. f satisfies layers 1 and 2 properties if and only if f satisfies LO, INC, GU and GD.

3. f satisfies layers 1, 2 and 3 properties if and only if f satisfies LO, INC, GU, GD and SGB.

For each of the properties GD, GU and SGB we present now necessary and sufficient conditions in terms of maximal $\mathcal{D}$-chains.

Proposition 2.5. *f satisfies GD iff for every nonempty chain $\mathcal{D} \subseteq Spec(S)$ and for every nonempty maximal $\mathcal{D}$-chain, $\mathcal{C} \subseteq Spec(R)$, the following holds: for every $P \in \mathcal{D}$ there exists $Q \in \mathcal{C}$ such that $f^{-1}[Q] \subseteq P$.*

Proof. ($\Rightarrow$) Let $P \in \mathcal{D}$. Note that $f^{-1}[Q] \in D$ for every $Q \in \mathcal{C}$ and since $\mathcal{D}$ is a chain, for every $Q \in \mathcal{C}$ we have $f^{-1}[Q] \subseteq P$ or $P \subseteq f^{-1}[Q]$. Assume to the contrary that $P \subset f^{-1}[Q]$ for all $Q \in \mathcal{C}$. Thus,

$$P \subseteq \bigcap_{Q \in \mathcal{C}} (f^{-1}[Q]) = f^{-1}[\bigcap_{Q \in \mathcal{C}} Q].$$

Obviously, $\bigcap_{Q \in \mathcal{C}} Q \subseteq Q$ for every $Q \in \mathcal{C}$ and $\bigcap_{Q \in \mathcal{C}} Q \in Spec(R)$ (note that $\mathcal{C}$ is not empty). Therefore, it is impossible that $P = f^{-1}[\bigcap_{Q \in \mathcal{C}} Q]$, since then $\mathcal{C} \cup \{\bigcap_{Q \in \mathcal{C}} Q\}$ is a $\mathcal{D}$-chain strictly containing $\mathcal{C}$.

So, we may assume that $P \subset f^{-1}[\bigcap_{Q \in \mathcal{C}} Q]$. Now, $\bigcap_{Q \in \mathcal{C}} Q$ is a prime ideal of R lying over $f^{-1}[\bigcap_{Q \in \mathcal{C}} Q] \in Spec(S)$ (note that $f^{-1}[\bigcap_{Q \in \mathcal{C}} Q] \subseteq f^{-1}[Q]$ for all $Q \in \mathcal{C}$ and $\mathcal{C}$ is a nonempty $\mathcal{D}$-chain; thus $f^{-1}[\bigcap_{Q \in \mathcal{C}} Q] \neq S$). Therefore by GD, there exists a prime ideal $Q' \subset \bigcap_{Q \in \mathcal{C}} Q$ which is lying over P. So, we get a $\mathcal{D}$-chain $\mathcal{C} \cup \{Q'\}$ that strictly contains $\mathcal{C}$, which is again impossible.

($\Leftarrow$) Let $P_1 \subset P_2 \in Spec(S)$ and let $Q_2 \in Spec(R)$ lying over P_2. Assume to the contrary that there is no prime ideal $Q_1 \subset Q_2$ lying over P_1. Let $\mathcal{C}$ denote a maximal $\{P_1, P_2\}$-chain containing Q_2. Hence, for all $Q \in \mathcal{C}$, $P_1 \subset f^{-1}[Q] (= P_2)$, a contradiction. $\square$

We shall now prove the dual of Proposition 2.5.

Proposition 2.6. *f satisfies GU iff for every nonempty chain $\mathcal{D} \subseteq Spec(S)$ and for every nonempty maximal $\mathcal{D}$-chain, $\mathcal{C} \subseteq Spec(R)$, the following holds: for every $P \in \mathcal{D}$ there exists $Q \in \mathcal{C}$ such that $P \subseteq f^{-1}[Q]$.*

Proof. ($\Rightarrow$) Let $P \in \mathcal{D}$ and assume to the contrary that $P \supset f^{-1}[Q]$ for all $Q \in \mathcal{C}$. Then,

$$P \supseteq \bigcup_{Q \in \mathcal{C}} (f^{-1}[Q]) = f^{-1}[\bigcup_{Q \in \mathcal{C}} Q].$$

Note that $Q \subseteq \bigcup_{Q \in \mathcal{C}} Q$ for every $Q \in \mathcal{C}$ (although $\bigcup_{Q \in \mathcal{C}} Q$ is not necessarily a prime ideal of R). Now, by Lemma 1.7, there exists a minimal prime ideal Q' over $\bigcup_{Q \in \mathcal{C}} Q$ and by Proposition 1.6, $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]) = f^{-1}[Q']$, which is a prime ideal of S (note that $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q] \subseteq P$, so one cannot have $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]) = S$).

Now, if $P = f^{-1}[\bigcup_{Q \in \mathcal{C}} Q])$ then $\mathcal{C} \cup \{Q'\}$ is a $\mathcal{D}$-chain strictly containing $\mathcal{C}$, a contradiction. If $f^{-1}[\bigcup_{Q \in \mathcal{C}} Q]) \subset P$ then by GU, there exists a prime ideal of R, $Q' \subset Q''$ lying over P. So, we get a $\mathcal{D}$-chain, $\mathcal{C} \cup \{Q''\}$ strictly containing $\mathcal{C}$, which is again impossible.

($\Leftarrow$) The proof is almost identical to the proof of ($\Leftarrow$) in Proposition 2.5. $\square$

Proposition 2.7. *f satisfies SGB iff for every nonempty chain $\mathcal{D} \subseteq Spec(S)$, for every nonempty maximal $\mathcal{D}$-chain, $C \subseteq Spec(R)$, and for every cut $\mathcal{A}$ of $\mathcal{C}$ such that $\mathcal{A} \neq (\emptyset, \mathcal{C})$ and $\mathcal{A} \neq (\mathcal{C}, \emptyset)$ (thus $|\mathcal{C}| \geq 2$), the following holds: for every $P \in \mathcal{D}$ there exists $Q_1 \in \mathcal{A}^L$ such that $P \subseteq f^{-1}[Q_1]$ or there exists $Q_2 \in \mathcal{A}^R$ such that $f^{-1}[Q_2] \subseteq P$.*

Proof. ($\Rightarrow$) Assume to the contrary that there exists $P \in \mathcal{D}$ such that

$$f^{-1}[Q_l] \subset P \subset f^{-1}[Q_r]$$

for all $Q_l \in \mathcal{A}^L$ and $Q_r \in \mathcal{A}^R$. Thus,

$$f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q] = \bigcup_{Q \in \mathcal{A}^L} f^{-1}[Q] \subseteq P \subseteq \bigcap_{Q \in \mathcal{A}^R} f^{-1}[Q] = f^{-1}[\bigcap_{Q \in \mathcal{A}^R} Q].$$

Now, $\bigcap_{Q \in \mathcal{A}^R} Q \in \mathrm{S}pec(R)$ and $\mathcal{C} \cup \{\bigcap_{Q \in \mathcal{A}^R} Q\}$ is a chain of prime ideals of R. Thus, since $\mathcal{C}$ is a maximal $\mathcal{D}$-chain, $\bigcap_{Q \in \mathcal{A}^R} Q$ is not lying over P. Hence,

$$P \subset f^{-1}[\bigcap_{Q \in \mathcal{A}^R} Q)].$$

On the other hand, $\bigcup_{Q \in \mathcal{A}^L} Q$ is an ideal of R lying over $f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q]$ and is contained in the prime ideal $\bigcap_{Q \in \mathcal{A}^R} Q$. Thus, by Lemma 1.7, there exists a minimal prime ideal $\bigcup_{Q \in \mathcal{A}^L} Q \subseteq Q' \subseteq \bigcap_{Q \in \mathcal{A}^R} Q$ over $\bigcup_{Q \in \mathcal{A}^L} Q$, and by Proposition 1.6, Q' is lying over $f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q] \in \mathrm{S}pec(S)$. Now, since $\mathcal{C} \cup \{Q'\}$ is a chain of prime ideals and $\mathcal{C}$ is a maximal $\mathcal{D}$-chain, one cannot have Q' lying over P. Thus,

$$f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q] \subset P.$$

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So we have a chain of prime ideals $f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q] \subset P \subset f^{-1}[\bigcap_{Q \in \mathcal{A}^R} Q)]$ in $\mathrm{S}pec(S)$ (note that $f^{-1}[\bigcap_{Q \in \mathcal{A}^R} Q)] \neq S$); we have $Q' \subset \bigcap_{Q \in \mathcal{A}^R} Q \in \mathrm{S}pec(R)$ lying over $f^{-1}[\bigcup_{Q \in \mathcal{A}^L} Q]$ and $f^{-1}[\bigcap_{Q \in \mathcal{A}^R} Q]$ respectively, such that $\mathcal{C} \cup \{Q', \bigcap_{Q \in \mathcal{A}^R} Q\}$ is a chain in $\mathrm{S}pec(R)$. Thus, by SGB, there exists $Q'' \in \mathrm{S}pec(R)$ such that

$$Q' \subset Q'' \subset \bigcap_{Q \in \mathcal{A}^R} Q$$

and $f^{-1}[Q''] = P$. But then $\mathcal{C} \cup \{Q''\}$ is a $\mathcal{D}$-chain strictly containing $\mathcal{C}$, a contradiction.

($\Leftarrow$) Let $P_1 \subset P_2 \subset P_3 \in \mathrm{S}pec(S)$ and let $Q_1 \subset Q_3 \in \mathrm{S}pec(R)$ such that Q_1 is lying over P_1 and Q_3 is lying over P_3. Assume to the contrary that there is no prime ideal $Q_1 \subset Q_2 \subset Q_3$ lying over P_2. Let $\mathcal{C}$ denote a maximal $\{P_1, P_2, P_3\}$-chain containing $\{Q_1, Q_3\}$. Let $\mathcal{A}$ be the cut defined by $\mathcal{A}^L = \{Q \in \mathcal{C} \mid f^{-1}[Q] = P_1\}$ and $\mathcal{A}^R = \{Q \in \mathcal{C} \mid f^{-1}[Q] = P_3\}$ (note that, by our assumption, for all $Q \in \mathcal{C}, f^{-1}[Q] \neq P_2$). Note that $\mathcal{A}^L \neq \emptyset$ and $\mathcal{A}^R \neq \emptyset$. Hence, $f^{-1}[Q_l] \subset P_2 \subset f^{-1}[Q_r]$ for all $Q_l \in \mathcal{A}^L$ and $Q_r \in \mathcal{A}^R$, a contradiction. □

Definition 2.8. Let $\mathcal{D} \subseteq \mathrm{S}pec(S)$ be a chain of prime ideals and let $\mathcal{C} \subseteq \mathrm{S}pec(R)$ be a $\mathcal{D}$-chain. We say that $\mathcal{C}$ is a *maximal cover* of $\mathcal{D}$ if $\mathcal{C}$ is a cover of $\mathcal{D}$ which is maximal with respect to containment.

Theorem 2.9. *f satisfies GD, GU and SGB if and only if for every nonempty chain $\mathcal{D} \subseteq Spec(S)$ and for every nonempty maximal $\mathcal{D}$-chain $\mathcal{C} \subseteq Spec(R)$, $\mathcal{C}$ is a maximal cover of $\mathcal{D}$.*

Proof. ($\Rightarrow$) Let $\mathcal{D} \subseteq \mathrm{S}pec(S)$ be a nonempty chain and let $\mathcal{C} \subseteq \mathrm{S}pec(R)$ be a nonempty maximal $\mathcal{D}$-chain. Assume to the contrary that $\mathcal{C}$ is not a cover of $\mathcal{D}$. So there exists a prime ideal $P \in \mathcal{D} \setminus \{f^{-1}[Q]\}_{Q \in \mathcal{C}}$. Obviously $\{f^{-1}[Q]\}_{Q \in \mathcal{C}} \cup \{P\}$ is a chain. We have the following three possibilities:

1. $P \subset f^{-1}[Q]$ for all $Q \in \mathcal{C}$. However, by assumption f satisfies GD and thus by Proposition 2.5, this situation is impossible.

2. $f^{-1}[Q] \subset P$ for all $Q \in \mathcal{C}$. However, by assumption f satisfies GU and thus by Proposition 2.6, this situation is impossible.

3. There exist $Q', Q'' \in \mathcal{C}$ such that $f^{-1}[Q'] \subset P \subset f^{-1}[Q'']$. So, let $\mathcal{A}$ be the cut defined by $\mathcal{A}^L = \{Q \in \mathcal{C} \mid f^{-1}[Q] \subset P\}$ and $\mathcal{A}^R = \{Q \in \mathcal{C} \mid P \subset f^{-1}[Q]\}$. Note that $\mathcal{A}^L \neq \emptyset$ and $\mathcal{A}^R \neq \emptyset$. Thus $f^{-1}[Q_l] \subset P \subset f^{-1}[Q_r]$ for all $Q_l \in \mathcal{A}^L$ and $Q_r \in \mathcal{A}^R$. However, by assumption f satisfies SGB and thus by Proposition 2.7, this situation is impossible.

Finally, it is obvious that $\mathcal{C}$ is a maximal cover of $\mathcal{D}$, since $\mathcal{C}$ is a cover of $\mathcal{D}$ and a maximal $\mathcal{D}$-chain.

($\Leftarrow$) It is obvious. □

We note that by [KO, Proposition 4 and Corollary 11], if $S \subseteq R$ are commutative rings such that R satisfies GU over S, then for every chain of prime ideals $\mathcal{D} \subseteq \mathrm{S}pec(S)$ there exists a chain of prime ideals in $\mathrm{S}pec(R)$ covering it. However, if for example R does not satisfy GD over S, not **every** maximal $\mathcal{D}$-chain is a cover of $\mathcal{D}$. See example 1.8.

We shall now present one of the main results of this paper.

Theorem 2.10. *Assume that f satisfies LO, INC, GD, GU and SGB. Let $\mathcal{D}$ denote a chain of prime ideals in $Spec(S)$. Then there exists a perfect maximal cover of $\mathcal{D}$. Moreover, any maximal $\mathcal{D}$-chain is a perfect maximal cover.*

Proof. If $\mathcal{D}$ is empty then a maximal $\mathcal{D}$-chain must be empty and the assertion is clear. So, we may assume that $\mathcal{D}$ is not empty. f satisfies LO and thus by Lemma 2.2, there exists a nonempty maximal $\mathcal{D}$-chain. Now, let $\mathcal{C}$ be any maximal $\mathcal{D}$-chain. f satisfies GD, GU and SGB and therefore by Theorem 2.9, $\mathcal{C}$ is a maximal cover of $\mathcal{D}$. Finally, f satisfies INC; hence by Lemma 1.10, $\mathcal{C}$ is a perfect maximal cover of $\mathcal{D}$. $\qquad\square$

We can now present the main result of this paper. The following corollary closes the circle.

Corollary 2.11. *The following conditions are equivalent:*

1. f satisfies layers 1,2 and 3 properties.

2. f satisfies LO, INC, GD, GU and SGB.

3. For every chain $\mathcal{D} \subseteq Spec(S)$ and for every maximal $\mathcal{D}$-chain $\mathcal{C} \subseteq Spec(R)$, $\mathcal{C}$ is a perfect maximal cover of $\mathcal{D}$.

4. For every chain $\mathcal{D} \subseteq Spec(S)$ and for every maximal $\mathcal{D}$-chain $\mathcal{C} \subseteq Spec(R)$, $|C| = |D|$.

Proof. $(1) \Rightarrow (2)$: By Proposition 2.4. $(2) \Rightarrow (3)$: By Theorem 2.10. $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$ are obvious. $\qquad\square$

Example 2.12. Notation as in Example 1.16 and assume that the value monoid of w is cancellative. Then R satisfies LO, INC, GU, GD, and SGB over O_v. So, for every chain $\mathcal{D} \subseteq \mathrm{S}pec(O_v)$ and for every maximal $\mathcal{D}$-chain $\mathcal{C} \subseteq \mathrm{S}pec(R)$, $\mathcal{C}$ is a perfect maximal cover of $\mathcal{D}$.

We note that one can easily construct examples of f that satisfies some of the properties. We shall present now an example in which the cardinality of the set of all chains in $\mathrm{S}pec(S)$ that satisfy property 4 of Corollary 2.11 is equal to the cardinality of $P(\mathrm{S}pec(S))$, the power set of $\mathrm{S}pec(S)$, and still f does not satisfy the equivalent conditions of Corollary 2.11 (since property 4 is not fully satisfied).

Example 2.13. Let A be a valuation ring of a field F with $|\mathrm{Spec}(A)| = a$, an infinite cardinal. Let Q_0 denote the maximal ideal of A and assume that Q_0 has an immediate predecessor, namely, a prime ideal $Q_1 \subset Q_0$ such that there is no prime ideal between Q_0 and Q_1. Let $B = A_{Q_1}$, a valuation ring of F with maximal ideal Q_1. Consider A as a subring of B and let $f : A \to B$ be the map defined by $f(a) = a$ for all $a \in A$. It is well known that there is a bijective map $Q \to Q \cap A$ from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A) \setminus \{Q_0\}$. Let

$$T = \{\mathcal{D} \subseteq \mathrm{Spec}(A) \mid \text{ for every maximal } \mathcal{D}\text{-chain } \mathcal{C} \subseteq \mathrm{Spec}(B), \ |C| = |D|\};$$

It is not difficult to see that

$$P(\mathrm{Spec}(A)) \setminus T = \{\mathcal{E} \subseteq \mathrm{Spec}(A) \mid Q_0 \in \mathcal{E}, |\mathcal{E}| = n \text{ for some } n \in \mathbb{N}\}.$$

It is obvious that $|P(\mathrm{Spec}(A)) \setminus T| = a$ and thus $|T| = |P(\mathrm{Spec}(A))| = 2^a$. Note that f does not satisfy LO and GU; although it does satisfy INC, GD and SGB.

Finally, denote $T' = \{\mathcal{D} \subseteq \mathrm{Spec}(A) \mid \text{ for every maximal } \mathcal{D}\text{-chain } \mathcal{C} \subseteq \mathrm{Spec}(B), \ \mathcal{C} \text{ is a perfect maximal cover of } \mathcal{D}\}$. Then $T' \subset T$; in fact, $T' = P(\mathrm{Spec}(A) \setminus \{Q_0\})$. In other words, the set of chains in $\mathrm{Spec}(A)$ satisfying property 3 of corollary 2.11 is strictly contained in the set of chains in $\mathrm{Spec}(A)$ satisfying property 4 of corollary 2.11.

We close this paper by presenting some results regarding maximal chains in $\mathrm{Spec}(S)$. We shall need to assume that f is unitary.

Lemma 2.14. *Assume that f is unitary. Let $\mathcal{D} \subseteq Spec(S)$ be a maximal chain and let $\mathcal{C} \subseteq Spec(R)$ be a maximal cover of $\mathcal{D}$. Then $\mathcal{C}$ is a maximal chain in $Spec(R)$.*

Proof. Assume to the contrary that $\mathcal{C}$ is not a maximal chain in $\mathrm{Spec}(R)$. Then there exists a prime ideal Q' of R such that $\mathcal{C} \cup \{Q'\}$ is a chain of prime ideals of R strictly containing $\mathcal{C}$. However, since $\mathcal{C}$ is a cover of $\mathcal{D}$, we have $\mathcal{D} = \{f^{-1}[Q]\}_{Q \in \mathcal{C}}$ and since $\mathcal{C}$ is a maximal cover of $\mathcal{D}$, we have $f^{-1}[Q'] \notin \mathcal{D}$. Note that, since f is unitary, $f^{-1}[Q'] \in \mathrm{Spec}(S)$. Therefore,

$$\{f^{-1}[Q]\}_{Q \in \mathcal{C}} \cup \{f^{-1}[Q']\}$$

is a chain in $\mathrm{Spec}(S)$ strictly containing $\mathcal{D}$, a contradiction.

$\square$

We are now able to prove that, assuming f is unitary and satisfies GD, GU and SGB, if $\mathcal{D}$ is a maximal chain in $\mathrm{Spec}(S)$ and $\mathcal{C}$ is any maximal $\mathcal{D}$-chain, then $\mathcal{C}$ must be a maximal chain in $\mathrm{Spec}(R)$.

Corollary 2.15. *Assume that f is unitary and satisfies GD, GU and SGB. Let $\mathcal{D}$ be a maximal chain in $Spec(S)$ (so $\mathcal{D}$ is not empty) and let $\mathcal{C} \subseteq Spec(R)$ be a nonempty maximal $\mathcal{D}$-chain. Then $\mathcal{C}$ is a maximal cover of $\mathcal{D}$ and a maximal chain in $Spec(R)$.*

Proof. By Theorem 2.9, $\mathcal{C}$ is a maximal cover of $\mathcal{D}$. By Lemma 2.14, $\mathcal{C}$ is a maximal chain in $Spec(R)$.

□

Corollary 2.16. *Assume that f is unitary and satisfies LO, GD, GU and SGB. Let $\mathcal{D}$ denote a maximal chain of prime ideals in $Spec(S)$. Then there exists a maximal cover of $\mathcal{D}$ which is a maximal chain of prime ideals in $Spec(R)$.*

Proof. By Lemma 2.2 there exists a nonempty maximal $\mathcal{D}$-chain, and by Corollary 2.15 it is a maximal chain of prime ideals in $Spec(R)$, covering $\mathcal{D}$.

□

Note: it is easy to see that if f is not unitary then the results presented in 2.14, 2.15 and 2.16 are no longer valid.

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